PASCAL'S ARITHMETICAL TRIANGLE.

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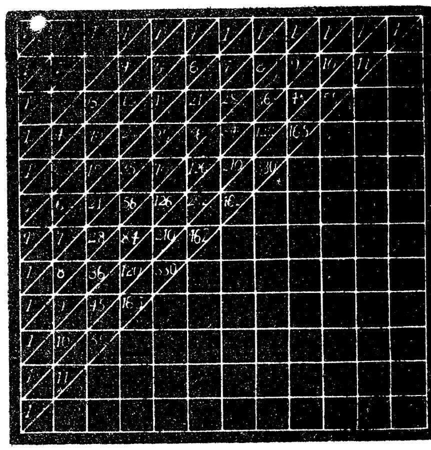
I do not remember of having ever seen an account of this interesting device in any of our American text books, and, so far as I am able to ascertain, it has not been published in this country. The accompaning diagram explaines itself.



To find any number, in a triangle, take the sum of the number immediately above and the number immediately to the left of the required number, or take the sum of the numbers immediately above and to the left of the required number. Thus, the 7th number in the 4th row $=28+56=84$, or

$$28+21+15+10+6+3+1=84.$$

The numbers on the diagonals are the coefficients of the expansion of a binomial.

The mth number in the nth row is given by the formula

$$\frac{|\underline{m+n-2}}{|\underline{(m-1)}\,|\underline{(n-1)}}.$$

Thus, the 7th number in the 5th row

$$=\frac{|\underline{7+\ -2}}{|\underline{(7-1)}\,|\underline{(5-1)}}=\frac{|\underline{10}}{|\underline{6}\ |\underline{4}}=\frac{1.2.3.4.5.6.7.8.9.10}{1.2.3.4.5.6.1.2.3.4}=210., \text{ etc.}$$

LOGICAL DEDUCTIONS FROM THE HYPOTHESIS THAT THE ANGLE-SUM IS LESS THAN TWO RIGHT ANGLES.

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Let ECF be any individual rectilineal triangle whatever whose angle sum is assumed to be less than two right angles.

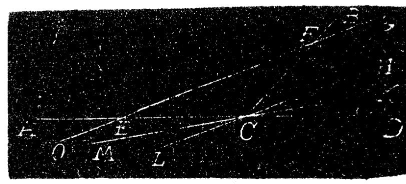
Produce the side EC to D. Then $ECF+FCD$=Two right angles. (Prop. XIII. Book I. of Euclid's Elements.) But by hypothesis $FEC+EFC+ECF<$Two right angles. Therefore $ECF+FCD>FEC+EFC+ECF$. Take ECF from both members.

Then in accordance with Euclid's axiom 5, $FCD>FEC+EFC$.

That is, one logical deduction from our hypothesis is that the exterior angle FCD is greater than the sum of the two interior and opposite angles FEC and EFC.

Lay off $FCH=EFC$.

Since $FCD>FCH$, CH must lie between CF and CD. Since FCH is equal to EFC, the line CH is parallel to the line EF according to proposition XXVII. Book I. of Euclid's Elements.



Euclid gives the following definition:

"Parallel straight lines are such as, being in the same plane, do not meet however far they are produced in both directions."

Since the lines CH and EF are parallel in the Euclidian sense they do *not* meet. Is the assumption *that there is a point* at which the two parallel lines meet "an extension" or a contradiction of Euclid's doctrine? Is it as difficult to accept as true Euclid's axiom 12 as the statement that there is a point at which two parallel straight lines meet? What Euclid calls his axiom 12 is regarded as a sound proposition although it may not be self-evident. *The statement that two parallel straight lines meet at a point called "infinity" is certainly not axiomatic and just acertainly has never been demonstrated.*

Again, lay off the angle $KCD=FEC$.

Since $KCD=FEC$, the line CK is parallel to the line EF according to Proposition XXVIII. Book I. of Euclid's Elements.

Since the lines CK and EF are parallel in the Euclidian sense they do *not* meet.

Since by inference from hypothesis $FCD>FEC+EFC$ and by construction $FCH+KCD=EFC+FEC$ it follows that $FCD>FCH+KCD$. But if FCD is greater than $FCH+KCD$, HCK must be a *real* angle.

It must be not only a *real* angle, but *an individual* angle; for according to the hypothesis from which we set out, the triangle ECF is an individual one, and hence HCK must be an individual angle.

It follows also that HCK, since it has but one value only for the same individual triangle ECF, can not be a variable. Further, if HCK is a real angle, it follows that HC and KC are seperate and distinct lines, although shown by logical deduction from the initial hypothesis to be parallel to the same line EF in the sense that they do not meet it when produced. The conclusion that the straight lines HC and KC meeting at C are both of them parallel to the straight line EF contradicts the statements known as Pleyfair's axiom that "Two straight lines, which intersect one another, can not be both parallel to the same straight line."

If the angle sum is two right angles, it readily follows that the ex-

terior angle FCD is equal to the sum of FCH and KCD and that the lines CH and CK are coincident. This deduction is contradicted by the conclusion that CH and CK are separate and distinct lines deduced from the hypothesis that the angle sum is less than two right angles.

Are the lines CH and CK the boundry lines between the *cutting* and the *not cutting* lines referred to in Lobatschewsky's theory of parallels, theorem 16?

CH as we have already seen is parallel to EF in the Euclidian sense; that is, these two lines will not meet however far they may be produced both ways.

CH can not be one of Lobatschewsky's boundary lines through C for they are supposed by him *gradually to approach and ultimately to meet EF.*

It was deduced from the hypothesis that CK, also, is parallel to EF in the Euclidian sense, and hence can not be one of Lobatschewsky's boundary lines through C.

If CH and CK are Lobatschewsky's boundary lines an inconsistency in his "Imaginary Geometry" emerges for his boundary lines through C ultimately meet EF whereas CH and CK can not do so without discrediting Propositions XXVII and XXVIII of Euclid's Elements. From this point of view Lobatschewsky seems to contradict inferences logically obtained from the hypothesis that the angle sum is less than two right angles.

Lobatschewsky's System of geometry is made up of two parts, one of which is distinctively Euclidian and the other, distinctively anti-Euclidian called by him "Imaginary Geometry". The "Imaginary Geometry" part flatly contradicts the Euclidian part. If CH and CK are Lobatcshewsky's boundary lines his "Imaginary Geometry" is seen to be inconsistent with itself.

The hypothesis of Lobatschewsky that the angle sum of a rectilineal triangle is less than two right angles contradicts the hypothesis of Euclid that it is equal to two right angles. According to the logical law of excluded middle, if one of two propositions that mutually contradict each other is true, the other must be false.

Let a straight line drawn in the plane of the triangle ECF through the point C be revolved about that point, will it occupy more than one position in which it does not cut the line EF?

The answer *no* is returned by those geometers who accept two right angles as the angle sum of a rectilineal triangle. With them CH and CK are one and the same straight line.

The answer *yes* must be the logical reply of those who accept less than two right angles as the angle sum of a rectilineal triangle and who also accept the first twenty-eight propositions of Book I., Euclid's Elements. With them CH and CK must be regarded as separate and distinct lines although neither is supposed to cut EF in any point whatever.

The answer *yes* is given by Lobatschewsky. His doctrine is that there is through C a pencil of not-cutting lines between two boundary lines that

untimately cut the line EF. These hypothetical boundary lines are called by him paralles.

These Lobatschewsky parallels are not Euclidian parallels, since according to Euclid's geometry but one straight line can be drawn through C in the plane of EF that does not cut EF.

Are the Lobatschewsky parallels in the same plane of those of Euclid or are they supposed to be on a pseudo-spherical surface? Lobatschewsky expressly says "rectilineal triangle." But a rectilineal triangle can not be drawn on a curved surface. Some of the followers of Lobatschewsky informs us, however, that the surface of Lobatschewsky's is not a Euclidian plane.

Is our Universe located in *one* space only or has Lobatschewsky discovered another and a different space to contain it — a space from which Euclid's 12th axiom is banished, where indeed the angle sum of a triangle is forever less than two right angles, where the strange device—"*Hypothesis Anguli Acuti*"—is emblazaned on all banners, where there are no planes but pseudo-spherical surfaces instead having negative curvature, and where there are no straight lines but lines lost eternally to rectitude.

For untold ages it has been believed that there is but *one* space, that in which the material bodies of the Universe are contained and through which they move in describing their orbits. This belief of the ages is contradicted by the Kantian Idealists who maintain that space is purely subjective, that is, that it is the product of and exists only in the human mind.

The mind of man cognizes but does not create space.

This belief of the ages referred to above is, also, discredited by Kantian agnostics and nihilists who doubt or deny the truth of the report made by our intelligence that material bodies occupy space and are contained in space There has from the outset been an antagonism between the science of Physics and Kantian idealism, Nihilism and agnosticism.

The space in which our universe is contained is currently believed to be trinally extended. By that is meant that three straight lines only mutually perpendicular to each other can be drawn through any point.

If space is *extended* it must be objective to the human mind.

In *objective space* straight and curved lines, planes and curved surfaces, and volumes such as cubes, cones and spheres may be located.

To call a line a space of one dimension; and a surface, a space of two dimensions is an absured use of language. Curved surfaces are found in space. To chartctorize *such surfaces* as *new species of space* is to subject the meaning of the word space to a strange, grotesque and irrational Metamorphosis.

We close this article with the remark that Euclid in his *Reductio ad absurdum* process argues from false premises in order to show that they are false and Lobatschewsky in his Theory of Parallels argues from false premises with such plausibility and subtle sophistry as to allure many into accepting both premises and conclusions as sound geometry.